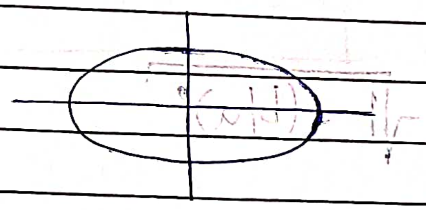
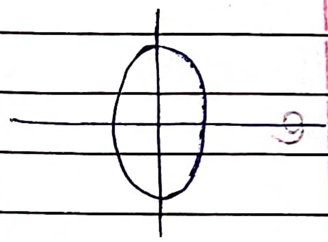
29/11/22

Ellipse

General Def^n. –

Locus of a pt. s.t. the ratio of its dist. from a fix. pt. & a fix. line is constant & less than 1.

Ellipse		
Eq^n	$x^2/a^2 + y^2/b^2 = 1$ $(a > b)$	$x^2/a^2 + y^2/b^2 = 1$ $(a < b)$
Centre	$(0,0)$	$(0,0)$
Foci	$(-ae, 0)$; $(ae, 0)$	$(0, -be)$; $(0, be)$
Directrices	$x = (-a/e)$; $x = a/e$	$y = (-b/e)$; $y = b/e$
Vertices	$(-a, 0)$; $(a, 0)$	$(0, -b)$; $(0, b)$
|Major Axis|	$2a$	$2b$

329

$$e = \sqrt{1 - \left(\frac{\text{Chota}}{\text{Bada}}\right)^2}$$

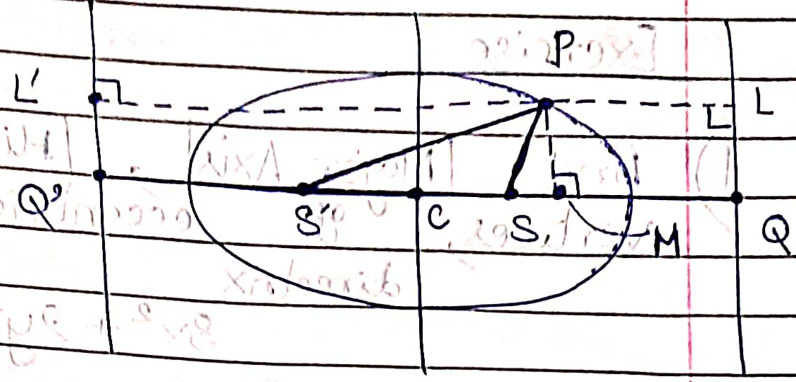
DATE
PAGE

|Minor Axis|	$2b$	$2a$
|L.R.|	$2b^2/a$	$2a^2/b$
L.R.	$x=(-ae)$; $x=ae$	$y=(-be)$; $y=be$
Endpts. of L.R.	$(-ae, -b^2/a)$ $(-ae, b^2/a)$ $(ae, -b^2/a)$ $(ae, b^2/a)$	$(-a^2/b, -be)$ $(a^2/b, -be)$ $(-a^2/b, be)$ $(a^2/b, be)$
e	$\sqrt{1-(b/a)^2}$	$\sqrt{1-(a/b)^2}$
Major Axis	$y=0$	$x=0$
Minor Axis	$x=0$	$y=0$

Second Def^n of Ellipse

Locus of pt. s.t. sum of its dist. from two fix. pts is const. & this const. is greater than dist. b/w the fix. pts.

Proof :



$PS = e \cdot PL$
$PS' = e \cdot PL'$

$\Rightarrow (PS + PS')$
$= e \cdot (PL + PL')$
$= e \cdot (MQ + MQ') = e \cdot (CQ - \cancel{CM} + CQ' + \cancel{CM})$
$= e \cdot (Q'C + CQ) = 2e \cdot QC = 2e \cdot a/e$

$\Rightarrow$ $\boxed{PS + PS' = 2a}$

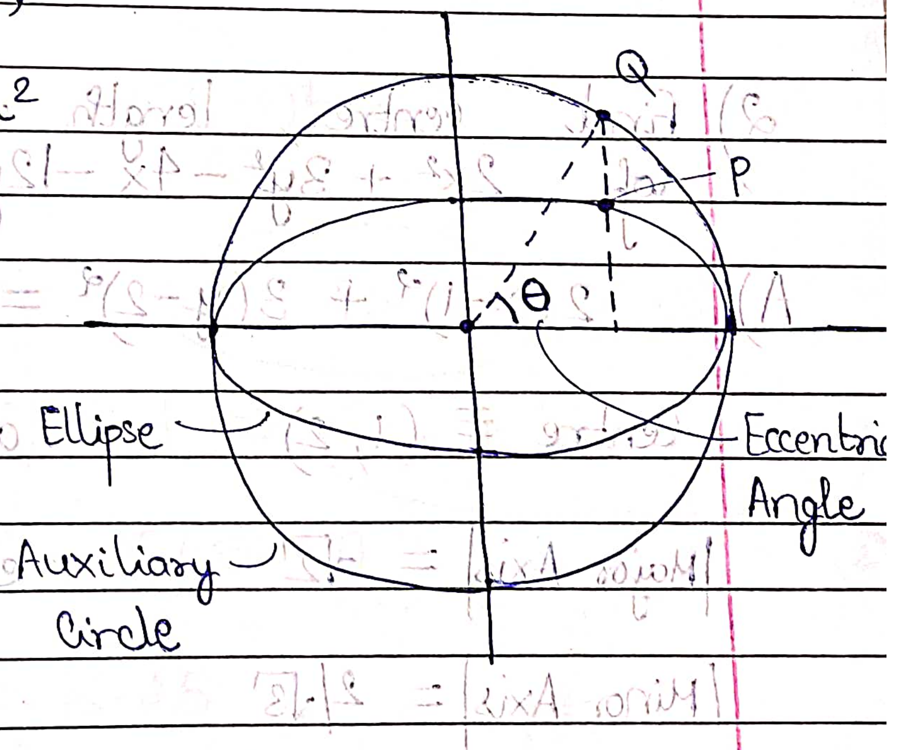
Parametric Coordinates

For horiz. ellipse,

Aux. ⊙ : $x^2 + y^2 = a^2$

P & Q are Corresponding pts.

$\boxed{P(a\cos(\theta), b\sin(\theta))}$



DATE
PAGE

Exercise

1) Find |Major Axis|, |Minor Axis|, foci, vertices, & eccentricity of directrix

$$3x^2+2y^2=6.$$

A) $a=\sqrt{2}$, $b=\sqrt{3}$, $e=1/\sqrt{3}$

Foci $\equiv$ $(0,-1)$; $(0,1)$

Directrices $\equiv$ $y=(-3)$; $y=3$

Vertices $\equiv$ $(0,-\sqrt{3})$; $(0,\sqrt{3})$

|Major axis| $=2\sqrt{3}$ |Minor axis| $=2\sqrt{2}$

2) Find centre, length of axes & eccentricity of $2x^2+3y^2-4x-12y+13=0$.

A) $2(x-1)^2+3(y-2)^2=1$

Centre $\equiv$ $(1,2)$ $a=1/\sqrt{2}$, $b=1/\sqrt{3}$

|Major Axis| $=\sqrt{2}$ $e=1/\sqrt{3}$

|Minor Axis| $=2/\sqrt{3}$

3) Find eqn of ellipse with foci (2, 3) & (−2, 3) whose semi minor axis is $\sqrt{5}$.

A) $b = \sqrt{5}$, Centre $\equiv (0, 3)$, $ae = 2$

$e^2 = 1 - (b/a)^2 \Rightarrow a = 3$

Eqn : $\frac{(x-0)^2}{3^2} + \frac{(y-3)^2}{(\sqrt{5})^2} = 1$

$\Rightarrow \boxed{x^2/9 + (y-3)^2/5 = 1}$

4) Find eqn of ellipse with centre (1, 2), one focus at (6, 2) & passing thru (4, 6)

A) Ellipse $\equiv \frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$

& $5 = ae \Rightarrow a^2 = 25 + b^2$ — ①

Also, $9/a^2 + 16/b^2 = 1 \Rightarrow 9b^2 + 16a^2 = a^2b^2$ — ②

① → ② $\Rightarrow 9b^2 + 16(25+b^2) = b^2(25+b^2)$

$\Rightarrow b^4 \cancel{15} = 16 \cdot 25$

$\Rightarrow b^2 = 20$ & $a^2 = 45$

333

DATE
PAGE

$\Rightarrow$ $\boxed{(x-1)^2/45 + (y-2)^2/20 = 1}$

5) If a chord joining 2 pts whose ecc. angle are α & β cuts the major axis of an ellipse at a dist. 'd' away from centre, then show that

$$\boxed{\tan(\alpha/2)\tan(\beta/2) = \left(\frac{d-a}{d+a}\right)}$$

A) Chord : $\left(\frac{x}{a}\right)c_{\left(\frac{\alpha+\beta}{2}\right)} + \left(\frac{y}{b}\right)s_{\left(\frac{\alpha+\beta}{2}\right)} = c_{\left(\frac{\alpha-\beta}{2}\right)}$

Thru $(d, 0)$ $\Rightarrow$ $\left(\frac{d}{a}\right) = \left(\frac{c_{\alpha/2}c_{\beta/2} + s_{\alpha/2}s_{\beta/2}}{c_{\alpha/2}c_{\beta/2} - s_{\alpha/2}s_{\beta/2}}\right)$

$\Rightarrow$ $\boxed{t_{\alpha/2}t_{\beta/2} = \left(\frac{d-a}{d+a}\right)}$

Formulae

1) Tangent (at pt. on curve) — $\boxed{T=0}$

2) Tangent (pt. outside curve) — $\boxed{SS_1 = T^2}$

3) Chord of Contact — $\boxed{T=0}$

4) Chord with given midpt — $\boxed{T=S_1}$

5) Eq^n of Tangent (at pt. on curve) —

$$(x_1, y_1) \Rightarrow \left(\frac{xx_1}{a^2}\right) + \left(\frac{yy_1}{b^2}\right) = 1$$

$$(a\cos\theta, b\sin\theta) \Rightarrow \left(\frac{x\cos\theta}{a}\right) + \left(\frac{y\sin\theta}{b}\right) = 1$$

6) Eq^n of Normal (at pt. on curve) —

$$(x_1, y_1) \Rightarrow \frac{(x-x_1)}{(x_1/a^2)} = \frac{(y-y_1)}{(y_1/b^2)}$$

DATE
PAGE

$$(a c_\theta,\ b s_\theta) \Rightarrow \left(\frac{ax}{c_\theta}\right) - \left(\frac{by}{s_\theta}\right) = (a^2 - b^2)$$

7) Tangent with given slope — $\boxed{y = mx \pm \sqrt{a^2m^2 + b^2}}$

Pts. of Contact — $\left(\frac{\mp a^2 m}{\sqrt{a^2m^2+b^2}},\ \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$

8) Cond^n for Tangent — $y = mx + c$ is tangent

$$\Updownarrow$$

$$c^2 = a^2m^2 + b^2$$

9) Director Circle — $\boxed{x^2 + y^2 = a^2 + b^2}$

10) Chord joining $P(\alpha)$ & $Q(\beta)$ —

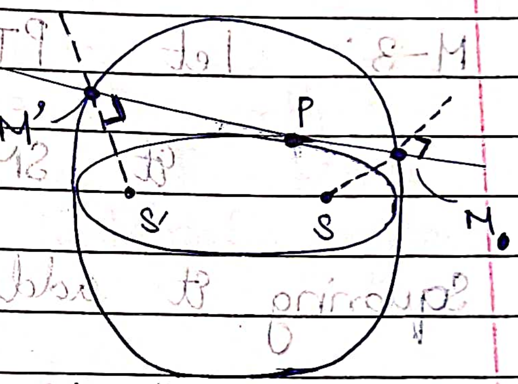
$$\boxed{\left(\frac{x}{a}\right)\cos\left(\frac{\alpha+\beta}{2}\right) + \left(\frac{y}{b}\right)\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)}$$

for focal chord, $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \left(\frac{e-1}{e+1}\right)$ or $\left(\frac{e+1}{e-1}\right)$

Props of Ellipse

1) Meeting pt. of $\perp$s from foci on any tangent to ellipse, lie on auxiliary $\odot$.

Proof – M-1 : $P \equiv (a\cos\theta, b\sin\theta)$



$\Rightarrow PT \equiv \left(\frac{x}{a}\right)\cos\theta + \left(\frac{y}{b}\right)\sin\theta = 1$

M : $\frac{x-ae}{(\cos\theta/a)} = \frac{y}{(\sin\theta/b)} = -\left(\frac{e\cos\theta - 1}{\cos^2\theta/a^2 + \sin^2\theta/b^2}\right) = \left(\frac{1}{1+e\cos\theta}\right)$

$\Rightarrow M \equiv \left(ae + \frac{\cos\theta}{a(1+e\cos\theta)}, \frac{\sin\theta}{a\sqrt{1-e^2}(1+e\cos\theta)}\right)$

Now, $\boxed{(x_M)^2 + (y_M)^2 = a^2}$

DATE
PAGE

M-2: In M-1, $PT \equiv (x/a)\cos\theta + (y/b)\sin\theta = 1$

& $SM \equiv \frac{x\sin\theta}{b} - \frac{y\cos\theta}{a} = \left(\frac{ae\sin\theta}{b}\right)$

Squaring & adding PT & SM gives

$x^2 + y^2 = a^2 \Rightarrow PT \cap SM$ lies on aux. ⊙

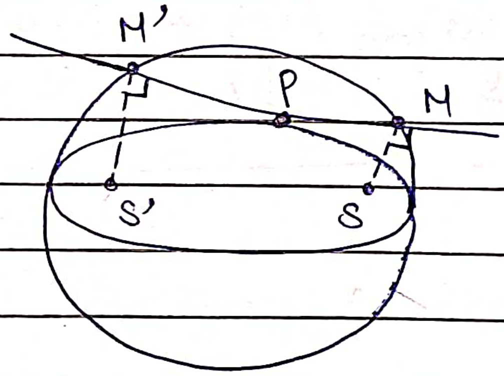
M-3: Let $PT \equiv y = mx \pm \sqrt{a^2m^2+b^2}$

& $SM \equiv my + x = ae$

Squaring & adding PT & SM gives

$x^2 + y^2 = a^2 \Rightarrow PT \cap SM$ lies on aux. ⊙.

2) Product of ⊥ dist. from foci on any tangent to ellipse is b^2



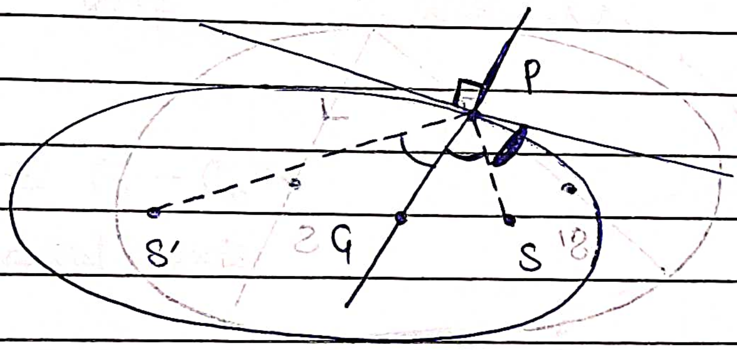
$SM \cdot S'M' = b^2$

Proof : $PT \equiv (x/a)\cos\theta + (y/b)\sin\theta = 1$

$SM = \frac{b(1-e\cos\theta)}{\sqrt{1-e^2\cos^2\theta}}$, $S'M' = \frac{b(1+e\cos\theta)}{\sqrt{1-e^2\cos^2\theta}}$

$\Rightarrow$ $SM \cdot S'M' = b^2$.

3) If S be focus & G be pt. where normal at P meets axis of ellipse, then $SG = e \cdot SP$ and tangent & normal at P bisects the external & internal angles b/w focal dist. of P.



Proof : $P \equiv (a\cos\theta, b\sin\theta) \Rightarrow PG \equiv \left(\frac{ax}{\cos\theta}\right) - \left(\frac{by}{\sin\theta}\right) = (a^2 - b^2)$

$\Rightarrow G \equiv (a\cos\theta\, e^2, 0)$ $\Rightarrow$ $SG = (ae - a\cos\theta\, e^2)$

Now, $SP = e\,PM = e\left(\frac{a}{e} - a\cos\theta\right) \Rightarrow SP = (a - a\cos\theta\, e)$

$\Rightarrow$ $SG = e \cdot SP$

339

DATE
PAGE

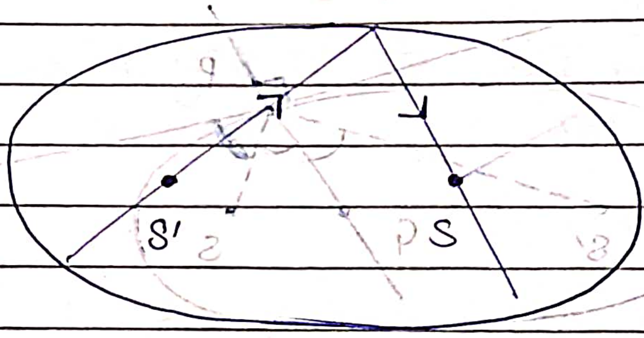
Similarly, $S'Q = e \cdot S'P$

By Angle Bisector Theorem, (PQ is $\angle$ bisector in $\triangle PSS'$)

$$\Rightarrow \angle S'PQ = \angle SPQ$$

4) An incoming light ray passes thru one foci of ellipse, then it will get reflected to other focus.

Proof: Using 3rd propt



5) If tangent at P meets directrix in F, then PF will subtend right angle at corresponding focus.

Proof: $PT \equiv (x/a)c_\theta + (y/b)s_\theta = 1$

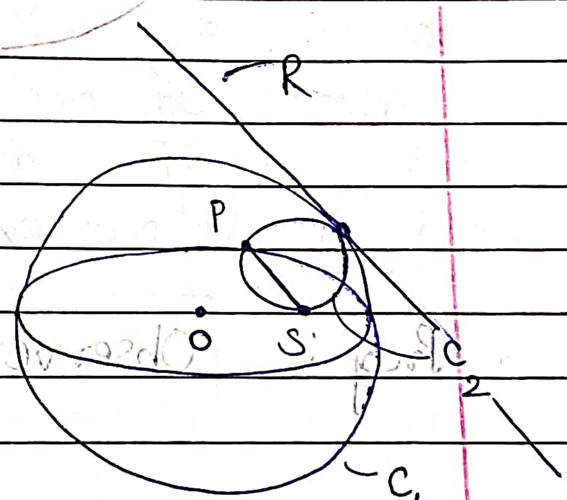
$f \equiv \left(\frac{a}{e}, \left(\frac{b}{es_\theta}\right)(e - c_\theta)\right)$; $P \equiv (ac_\theta, bs_\theta)$; $S \equiv (ae, 0)$

$m_{PS} \cdot m_{SF} = \frac{bs_\theta}{a(c_\theta - e)} \cdot \frac{(b)(e-c_\theta)}{(es_\theta)\left(\frac{a}{e} - ae\right)} = (-1)$

$\Rightarrow \angle PSF = 90^\circ$

6) Tangents at extremities of L.R. $\cap$ on corresponding directrix.

7) The $\odot$ on any focal dist. as diameter touches the aux. $\odot$.



Proof: $R = C_1 - C_2$
(radical axis

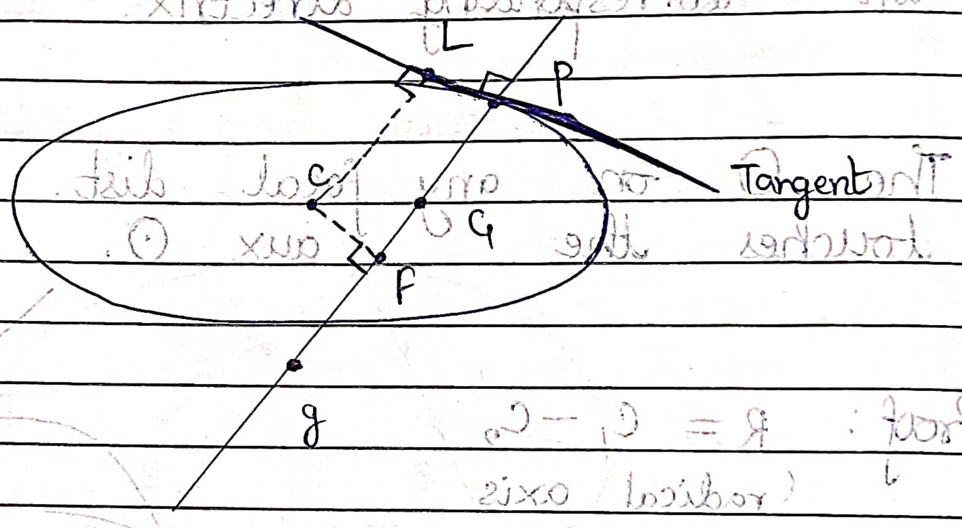
$\Rightarrow R \equiv a(e + c_\theta)x + (bs_\theta)y = (a^2ec_\theta)$

If dist. b/w O & R = a
$\Rightarrow$ Radical axis is tangent $\Rightarrow$ $\odot$s touch.

$d = \left|\frac{a^2ec_\theta}{\sqrt{a^2(e+c_\theta)^2 + b^2s_\theta^2}}\right| = \left|\frac{a^2ec_\theta}{\sqrt{a^2 + 2a^2ec_\theta + a^2(ec_\theta)^2}}\right| = \left|\frac{aec_\theta}{ec_\theta + 1}\right|$

8) If normal at any pt. P meets major axis and minor axis at G and g resp., and if CF is ⊥ upon this normal from centre C, then

$$\boxed{PF \cdot PG = b^2} \quad \& \quad \boxed{PF \cdot Pg = a^2}$$



Proof : Observe $PF = \left(\begin{array}{c} \perp \text{ dist. from } C \\ \text{on tangent at } P \end{array} \right)$

$PT \equiv (x/a)c_\theta + (y/b)s_\theta = 1$

$PG \equiv Pg \equiv (ax/c_\theta) - (by/s_\theta) = (a^2 - b^2)$

$$Pf = CL = \frac{1}{\sqrt{\left(\frac{c_\theta}{a}\right)^2 + \left(\frac{s_\theta}{b}\right)^2}} = \frac{b}{\sqrt{1-(ec_\theta)^2}}$$

$$G \equiv (ae^2 c_\theta, 0) \qquad g \equiv (0, (-a^2/b) s_\theta e^2)$$

$$PG = \sqrt{(ac_\theta)^2(1-e^2)^2 + (bs_\theta)^2} = b\sqrt{1-(ec_\theta)^2}$$

9) In general, 4 normals can be drawn from any pt. onto ellipse. If $\alpha, \beta, \gamma, \delta$ be eccentric angles of conormal pts, then

$$\boxed{\alpha + \beta + \gamma + \delta = (2n+1)\pi}$$

M-1

Proof: Normal $\equiv (ax/c_\theta) - (by/s_\theta) = (a^2 - b^2)$

This pass thru $(h, k) \Rightarrow \left(\frac{ah}{c_\theta}\right) - \left(\frac{bk}{s_\theta}\right) = (a^2 - b^2)$

Create 4th degree eq^n in $t_{(\theta/2)}$

$\Rightarrow$ It has 4 roots.

By trignometry show, $\Sigma\alpha = (2n+1)\pi$

by using $\tan\left(\Sigma(\alpha/2)\right) = 0$

M-2: Let 'z' be pt. on aux. circle with arg (θ). Assume aux. $\odot$ $\equiv$ unit $\odot$

$$\Rightarrow \quad c_\theta = \left(\frac{z^2+1}{2z}\right) \quad \& \quad s_\theta = \left(\frac{z^2-1}{2zi}\right)$$

Now, $ax\,S_\theta - by\,C_\theta = a^2-b^2$ is a normal to ellipse. Let it pass thru fix (h,k)

$$\Rightarrow \quad \frac{ah}{\left(\frac{z^2+1}{2z}\right)} - \frac{bk}{\left(\frac{z^2-1}{2iz}\right)} = (a^2-b^2)$$

$$\Rightarrow \quad (ah)\left(\frac{z^2-1}{2zi}\right) - (bk)\left(\frac{z^2+1}{2z}\right) = (a^2-b^2)\left(\frac{z^4-1}{4z^2 i}\right)$$

$$\Rightarrow \quad (ah)(2z)(z^2-1) - (bki)(2z)(z^2+1)$$

$$= (a^2-b^2)(z^4-1)$$

$$\Rightarrow z^4\left[a^2-b^2\right]+z^3\left[2bki-2ah\right]+z\left[2bki+2ah\right]+\left[b^2-a^2\right]=0$$

This is 4th degree in z.

$\Rightarrow$ It has 4 roots. $\Rightarrow \exists$ 4 normals thru (h,k)

Let z_1, z_2, z_3, z_4 be roots.

Let they be corresponding pts of pts on ellipse, each with eccentric angles $\alpha, \beta, \gamma, \delta$.

$\Rightarrow \arg(z_1)=\alpha,\ \arg(z_2)=\beta,\ \arg(z_3)=\gamma,\ \arg(z_4)=\delta.$

By quartic eq^n, $z_1z_2z_3z_4=(-1)$

$$\Rightarrow e^{i(\alpha+\beta+\gamma+\delta)}=e^{i(2n+1)\pi}$$

$$\Rightarrow \alpha+\beta+\gamma+\delta=(2n+1)\pi$$

DATE

PAGE

10) If α, β, γ are eccentric angles of 3 pts. on ellipse, the normals at which are concurrent then

$$\boxed{\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0}$$

Proof: In prev. method, we had

$$\sum z_1 z_2 = 0 \Rightarrow \boxed{e^{i(\alpha+\beta)} + e^{i(\beta+\gamma)} + e^{i(\gamma+\delta)} + e^{i(\delta+\alpha)} + e^{i(\alpha+\gamma)} + e^{i(\beta+\delta)}} = 0$$

$$\Rightarrow s_{(\alpha+\beta)} + s_{(\beta+\gamma)} + s_{(\gamma+\alpha)} + s_{(\gamma+\delta)} + s_{(\delta+\alpha)} + s_{(\beta+\delta)} = 0$$

By prev. propt, $\alpha+\beta+\gamma+\delta = (2n+1)\pi$

$$\Rightarrow s_{(\alpha+\beta)} = \sin\left((2n+1)\pi - (\gamma+\delta)\right)$$

$$\Rightarrow s_{(\alpha+\beta)} = s_{(\gamma+\delta)}$$

$$s_{(\alpha+\beta)} + s_{(\beta+\gamma)} + s_{(\gamma+\alpha)} + s_{(\alpha+\beta)} + s_{(\beta+\gamma)} + s_{(\alpha+\gamma)} = 0$$

$$\Rightarrow (2)\left[s_{(\alpha+\beta)} + s_{(\beta+\gamma)} + s_{(\gamma+\alpha)}\right] = 0$$

$$\Rightarrow s_{(\alpha+\beta)} + s_{(\beta+\gamma)} + s_{(\gamma+\alpha)} = 0$$

In general ellipse,

$$A < B \iff \frac{\left(\text{Dist. from Major Axis}\right)^2}{A^2} + \frac{\left(\text{Dist. from Minor Axis}\right)^2}{B^2} = 1$$

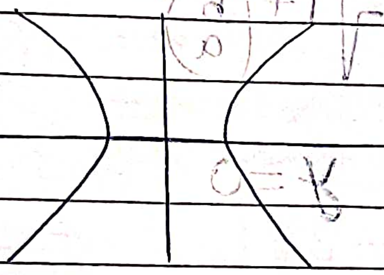
$$A > B \iff \frac{\left(\text{Dist. from Minor Axis}\right)^2}{A^2} + \frac{\left(\text{Dist. from Major Axis}\right)^2}{B^2} = 1$$

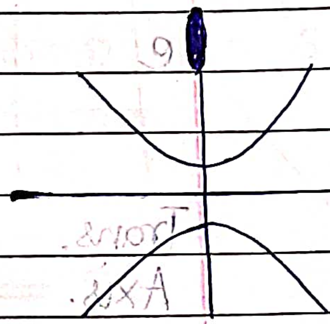
Hyperbola

General Def^n —

Locus of a pt. s.t. ratio of its dist. from a fix. pt. & a fix. line is constant & greater than 1.

Hyperbola





Eq^n	$x^2/a^2 - y^2/b^2 = 1$	$y^2/b^2 - x^2/a^2 = 1$
Centre	$(0,0)$	$(0,0)$
Foci	$(-ae,0)$; $(ae,0)$	$(0,-be)$; $(0,be)$
Directrices	$x = (-a/e)$; $x = a/e$	$y = (-b/e)$; $y = b/e$
Vertices	$(-a,0)$; $(a,0)$	$(0,-b)$; $(0,b)$
Transverse Axis	$2a$	$2b$
Conjugate Axis	$2b$	$2a$

357

$$e = \sqrt{1+\left(\frac{-ve}{+ve}\right)^2}$$

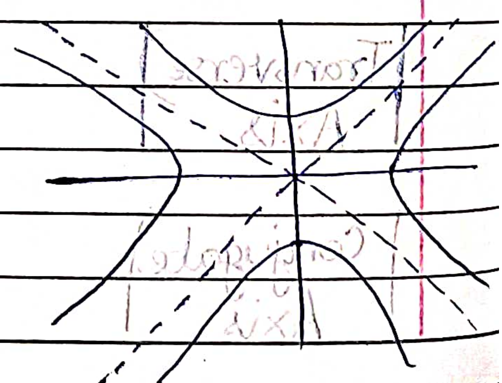
DATE
PAGE

$\lvert L.R. \rvert$	$2b^2/a$	$2a^2/b$
L.R.	$x=(-ae)$; $x=ae$	$y=(-be)$; $y=be$
Endpts of L.R	$(-ae, -b^2/a)$ $(-ae, b^2/a)$ $(ae, -b^2/a)$ $(ae, b^2/a)$	$(-a^2/b, -be)$ $(a^2/b, -be)$ $(-a^2/b, be)$ $(a^2/b, be)$
e	$\sqrt{1+\left(\frac{b}{a}\right)^2}$	$\sqrt{1+\left(\frac{a}{b}\right)^2}$
Trans. Axis.	$y=0$	$x=0$
Conj. Axis.	$x=0$	$y=0$

<u>Conjugate Hyperbola</u>

Hyperbola obtained by interchanging trans. & conj. axis of given hyperbola

Hyperbola : $(x/a)^2-(y/b)^2=1$

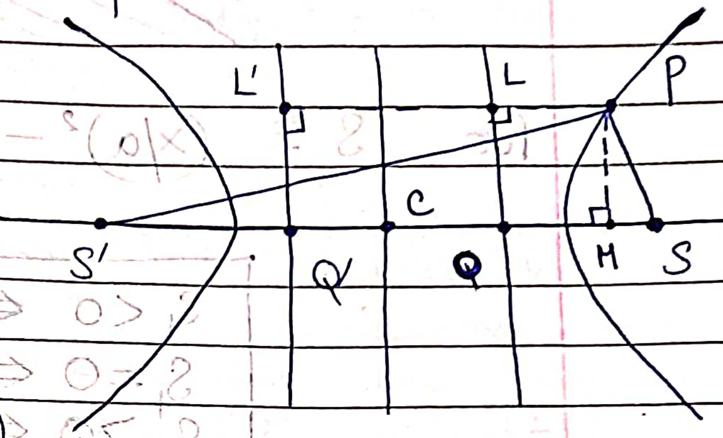


Conj. Hyperbola : $(x/a)^2-(y/b)^2=(-1)$

Second Def^n of Hyperbola

Locus of pt. s.t. diff. of its dist. from two fix pts. is const & this const. is smaller than dist. b/w the fix pts.

Proof: $PS = e \cdot PL$
$PS' = e \cdot PL'$



$\Rightarrow (PS' - PS)$
$= e \cdot (PL' - PL)$
$= e \cdot (MQ' - MQ)$
$= e \cdot (MC + CQ' - MC + CQ')$
$= 2e \cdot CQ' \Rightarrow$

$$\boxed{|PS' - PS| = 2a}$$

Parametric Coordinates

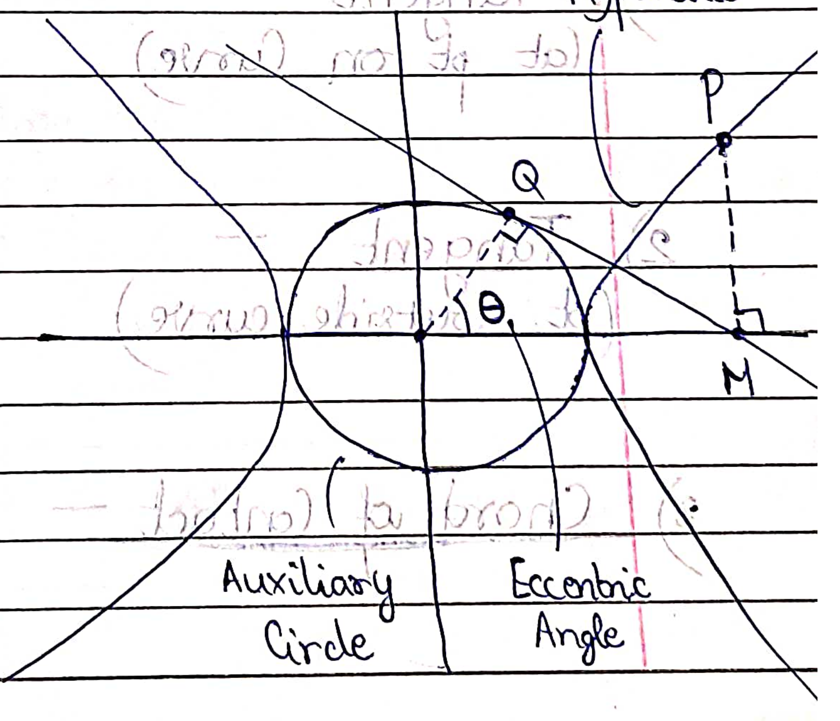
For horiz. hyperbola,

$$\boxed{P \equiv (a\sec_\theta, b\tan_\theta)}$$

$Q \equiv (a\cos_\theta, b\sin_\theta)$

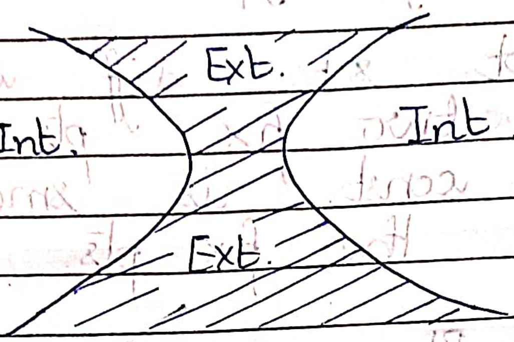
$QM \equiv x\cos_\theta + y\sin_\theta = a$

$\Rightarrow M(a\sec_\theta, 0)$



DATE
PAGE

Posit. of Pt. wrt)(



for $S \equiv (x/a)^2 - (y/b)^2 = 1$,

$S_1 < 0 \iff$ Pt. OUTSIDE
$S_1 = 0 \iff$ Pt. ON CURVE
$S_1 > 0 \iff$ Pt. INSIDE

Formulae

1) Tangent (at pt on Curve) — $\boxed{T=0}$

2) Tangent (pt. outside curve) — $\boxed{SS_1 = T^2}$

3) Chord of Contact — $\boxed{T=0}$

4) Chord with given midpt. + $\boxed{T=S_1}$

5) Eqn of Tangent – (at pt. on curve)

$(x_1, y_1) \Rightarrow \left(\frac{xx_1}{a^2}\right) - \left(\frac{yy_1}{b^2}\right) = 1$

$(aS_\theta, bt_\theta) \Rightarrow \left(\frac{xS_\theta}{a}\right) - \left(\frac{yt_\theta}{b}\right) = 1.$

6) Eqn of Normal – (at pt. on curve)

$\left(\frac{a^2x}{x_1}\right) + \left(\frac{b^2y}{y_1}\right) = a^2+b^2$

$(x_1, y_1) \Rightarrow \cancel{\frac{(x-x_1)}{(x_1/a^2)} = \frac{(y-y_1)}{(y_1/(-b^2))}}$

$(aS_\theta, bt_\theta) \Rightarrow \left(\frac{ax}{S_\theta}\right) + \left(\frac{by}{t_\theta}\right) = a^2+b^2$

7) Tangent with given slope –

$\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$

8) Director ⊙ – $\boxed{x^2+y^2 = a^2-b^2}$

9) Chord joining $P(\alpha)$ & $Q(\beta)$ –

$$\boxed{\left(\frac{x}{a}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \left(\frac{y}{b}\right)\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)}$$

DATE

PAGE

Asymptotes

The st line, to which the tangent to the curve tends as the pt. of contact tends to approach ∞.

To find eqn of asymptotes of

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$$

Let $P(a\sec\theta, b\tan\theta)$ be a pt. on it.

$$PT \equiv \left(\frac{x}{a}\right)\sec\theta - \left(\frac{y}{b}\right)\tan\theta = 1$$

As $\theta \to \pm\pi/2$, $P \to \infty$.

$$PT \equiv \left(\frac{x}{a}\right) - \left(\frac{y}{b}\right)\sin\theta = \cos\theta$$

$(\theta \to \pm\pi/2)$

$$\left(\frac{x}{a}\right) - \left(\frac{y}{b}\right)(\pm 1) = 0 \Rightarrow \boxed{y = \left(\frac{\pm b}{a}\right)x}$$

Eqn of Asymptotes

Joint Eq^n of Asymptotes — $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0}$

1) In general, asymptotes of a given hyperbola pass thru its centre.

2) Angle b/w Asymptotes $= \tan^{-1}\left|\dfrac{2ab}{a^2-b^2}\right|$

3) Angle b/w asymptotes of Rect. Hyperbola $= 90°$

4) Eq^n of asymptotes is same for a given hyperbola & its conj. hyperbola.

5) Eq^n of hyperbola & joint eq^n of asymptotes differ only by a real const. term.

6) Any line ∥ to asymptote of hyperbola would meet the curve only at 1 pt.

7) Const. terms of eq^n of hyperbola, joint eq^n of asymptotes & eq^n of conj. hyperbola are in A.P.

Q) find asymptotes of $xy - 3y - 2x = 0$

A) Asymptotes : $xy - 2x - 3y + \lambda = 0$

$$\Delta = 0 \Rightarrow \begin{vmatrix} 0 & +1/2 & (-1) \\ +1/2 & 0 & (-3/2) \\ (-1) & (-3/2) & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & -3 \\ -2 & -3 & 2\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ -2 & -3 & 2\lambda-6 \end{vmatrix} = 0$$

$$\Rightarrow (2\lambda - 6) - 6 = 0 \Rightarrow \boxed{\lambda = 6}$$

Q) find hyp. with asymp. $2x - y = 3$ & $3x + y = 7$ which passes thru $(1,1)$

A) Asymp : $(2x - y - 3)(3x + y - 7) = 0$

$$\Rightarrow 6x^2 - y^2 - xy - 23x + 4y = (-21)$$

Since hyp. thru $(1,1)$

$\Rightarrow$ Hyp : $\boxed{6x^2 - y^2 - xy - 23x + 4y \; (+15) = 0}$

Q) Asymp. of hyp. with centre $(1,2)$ are $\parallel$ to $2x + 3y = 0$ & $3x + 2y = 0$. If hyp. thru $(5,3)$, find its eq^n.

A) Asymp : $(2x + 3y - 8)(3x + 2y - 7) = 0$

$$\Rightarrow \quad 6x^2 + 6y^2 + 13xy - 38x - 37y = (-56)$$

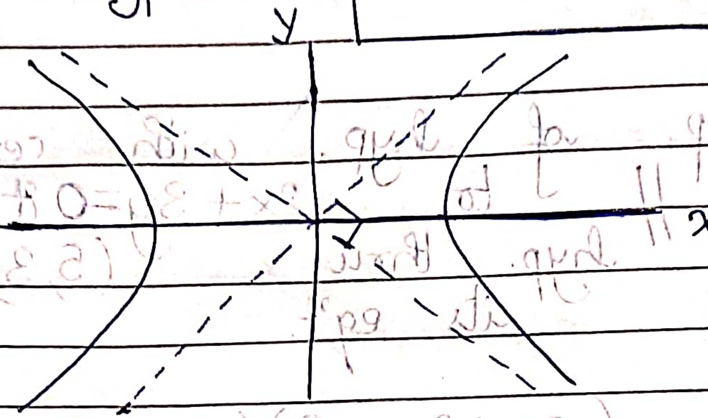
Since hyp. thru $(5,3)$.

$\Rightarrow$ Hyp : $\boxed{6x^2 + 6y^2 + 13xy - 38x - 37y = 98}$

DATE
PAGE

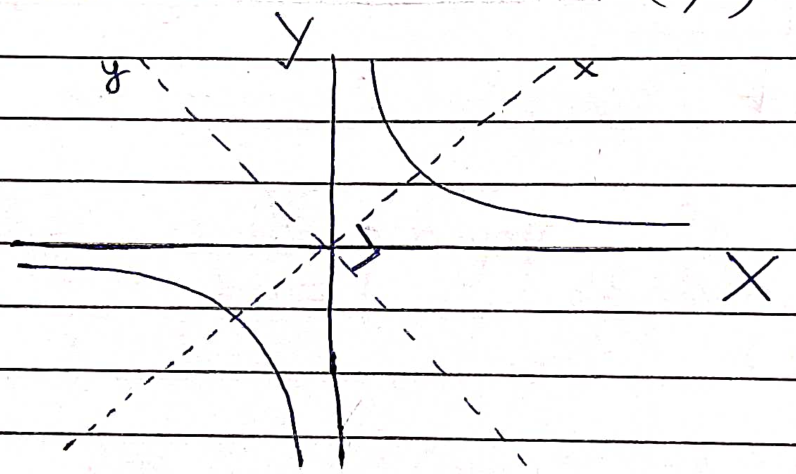
Rectangular Hyperbola

Eqⁿ of Rect. Hyp : $\boxed{x^2 - y^2 = a^2}$



Most popular form : $\boxed{xy = c^2}$ where $c^2 = \frac{a^2}{2}$

In above fig. Keep hyp. fix. & rotate axes about (0,0) by $-\pi/4$



Using, $(x+yi) = (X+Yi)e$

1) Parametric Coor : $\boxed{P \equiv (ct, c/t)}$

2) Tangent at $P \equiv \left(\frac{x(c/t) + y(ct)}{2}\right) = c^2$

$\Rightarrow$ $\boxed{\text{Slope of Tangent} = \left(\frac{-1}{t^2}\right)}$

3) Normal at $P \equiv (y - c/t) = t^2(x - ct)$

$\Rightarrow$ $\boxed{\text{Slope of Normal} = t^2}$

Q) Let A, B, C be vertices of Δ on $xy = c^2$ with $90°$ at B. P.t. tangent at B will be $\perp$ to ~~hyperbola~~ AC.

A) $A \equiv (ct_1, c/t_1)$; $C \equiv (ct_2, c/t_2)$

$B \equiv (ct_3, c/t_3)$

$\angle ABC = 90° \Rightarrow \left(\frac{1/t_2 - 1/t_3}{t_2 - t_3}\right)\left(\frac{1/t_1 - 1/t_3}{t_1 - t_3}\right) = (-1) \Rightarrow \boxed{t_1^2 = \left(\frac{-1}{t_2 t_3}\right)}$

Slope of AC = $\left(-1/t_2t_3\right)$ $\longrightarrow$ $\boxed{\text{Tangent} \perp \text{to AC}}$

Slope of tangent at B = $\left(-1/t_1^2\right)$ $\longrightarrow$

369

DATE
PAGE

Q) 1) Show that normals to $xy = c^2$ at pt. 't' meets curve again at pt. 't_1', then $t_1 t^3 = (-1)$

2) Normals at P, Q, R on $xy = c^2$ intersect at pt. on curve P.t. centre of hyp. is centroid of ΔPQR.

A) 1) $(ct, c/t) \Rightarrow$ Normal: $\left(y - \frac{c}{t}\right) = t^2(x - ct)$

Thru $(ct_1, c/t_1) \Rightarrow (c)\left(\frac{1}{t_1} - \frac{1}{t}\right) = t^2(c)(t_1 - t)$

$\Rightarrow \boxed{t_1 t^3 = (-1)}$

2) $P \equiv (ct_1, c/t_1)$, $Q \equiv (ct_2, c/t_2)$,

$R \equiv (ct_3, c/t_3)$.

Normal at P : $(y - c/t_1) = (t_1^2)(x - ct_1)$

Normal at Q : $(y - c/t_2) = (t_2^2)(x - ct_2)$

Normal at R : $(y - c/t_3) = (t_3^2)(x - ct_3)$

~~If this pass thru (h, k),~~

$$\cancel{ct^4 - ht^3 + kt - c = 0} \begin{cases} t_1 \\ t_2 \\ t_3 \\ t_4 \end{cases}$$

~~Now,~~

$$\cancel{\sum t_1 t_2 = (t_4)\Big(\overbrace{\textstyle\sum t_1}^{(t_1+t_2+t_3)}\Big) + (t_1t_2 + t_2t_3 + t_3t_1) = 0}$$

$$\cancel{\sum t_1t_2t_3 = (t_4)(t_1t_2 + t_2t_3 + t_3t_1) + t_1t_2t_3}$$

$$\cancel{\prod t_1 = (t_4)(t_1t_2t_3) = (-1)}$$

~~By~~ Let all these meet at $(ct_4, c/t_4)$.

By prev. Q,

$$t_4 t_1^3 = (-1)$$
$$t_4 t_2^3 = (-1)$$
$$t_4 t_3^3 = (-1)$$

$$\Rightarrow \quad t^3 = \left(-1/t_4\right) \begin{cases} t_1 \\ t_2 \\ t_3 \end{cases}$$

$\sum(t_1) = 0$ & $\sum(1/t_1) = 0$

$$\boxed{\text{Centroid} \equiv (0, 0)}$$

37)

Q) P.t. the locus of a pt. which moves s.t. sum of slopes of normals from it to $xy = c^2$ is equal to the sum of ordinates of foot of normals is a parabola.

A) Let pt be (h, k). Let feet of normals be $(ct_1, c/t_1)$; $(ct_2, c/t_2)$; $(ct_3, c/t_3)$; $(ct_4, c/t_4)$.

Normal, $(y - c/t) = (t^2)(x - ct)$

$$\Rightarrow ct^4 - ht^3 + kt - c = 0 \quad \begin{cases} t_1 \\ t_2 \\ t_3 \\ t_4 \end{cases}$$

We need, $\sum t_1^2 = \sum(1/t_1)$

$$\Rightarrow \left(\sum t_1\right)^2 = \underbrace{\cancel{\sum t_1^2}}_{\sum(1/t_1)} + 2\sum t_1 t_2$$

$$\Rightarrow (+h/c)^2 = \underbrace{\cancel{h/c}}_{\frac{\sum(t_1 t_2 t_3)}{\prod t_1}} + 0$$

$$\Rightarrow \left(\frac{h^2}{c^2}\right) = \frac{(-k/c)}{(-c/c)} \Rightarrow \boxed{h^2 = c^2 k}$$

Intersection of Circle & Rect. Hyp.

Consider a ⊙ $x^2+y^2+2gx+2fy+c=0$ and a rect. hyp. whose centre is (α, β), intersecting each other at 4 pts.

$A(x_1, y_1)$; $B(x_2, y_2)$; $C(x_3, y_3)$; $D(x_4, y_4)$

Then,

$$\left(\frac{\sum x_1}{4}\right) = \left(\frac{\alpha + (-g)}{2}\right) \quad \text{A.M. of}$$

$$\left(\frac{\sum y_1}{4}\right) = \left(\frac{\beta + (-f)}{2}\right)$$

i.e. $\boxed{\left(\text{A.M. of Pts.}\right) = \left(\text{A.M. of Centres}\right)}$

Proof: Let ⊙ : $x^2+y^2+2gx+2fy+c=0$
& hyp. : $xy = \lambda$.

$\cap \Rightarrow \quad x^2 + \lambda^2/x^2 + 2gx + 2f\lambda/x + c = 0$

$\Rightarrow \quad x^4 + 2gx^3 + \cancel{2f\lambda x}\, cx^2 + 2f\lambda x + \lambda^2 = 0$

$\Rightarrow \left(\sum x_1 = (-2g)\right)$ Similarly doing for 'y' gives.

$\Rightarrow \left(\sum y_1 = (-2f)\right)$

DATE
PAGE

Q) A rect. hyp. with centre C is ∩ by ⊙ with radius 'r' in 4 pts P, Q, R, S.

P.t. $CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2$

Q) A circle with centre $(3\alpha, 3\beta)$ and a variable radius cuts $x^2 - y^2 = 9a^2$ at P, Q, R, S.

P.t. locus of centroid of Δ PQS is

$$(x-2\alpha)^2 - (y-2\beta)^2 = a^2$$

Q) 5 pts. are given on a ⊙ of radius 'a'. Rect hyp.s are made thru 4 of these, one at a time

P.t. centre of hyp.s lie on a circle of radius 'a/2'

A) ~~Let hyp. $xy = c^2$. Let $C \equiv (\alpha, \beta)$~~

~~⊙ $x^2 + y^2$~~

Let hyp. $xy = \lambda^2$ & ⊙ be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Let pts. of $\cap$ be $(\lambda t_i, \lambda/t_i)$

$\Rightarrow$ $\text{Req.} = (\lambda^2)\left(\sum t_i^2 + \sum (1/t_i^2)\right)$

Now, $\lambda^2 t^2 + \dfrac{\lambda^2}{t^2} + 2g\lambda t + 2f\dfrac{\lambda}{t} + c = 0 — t_i$

$\Rightarrow$ $(\lambda^2)t^4 + (2g\lambda)t^3 + (c)t^2 + (2f\lambda)t + \lambda^2 = 0$ — t_i

$$\sum t_i^2 = \left(\sum t_i\right)^2 - 2\sum t_i t_j$$

$$= \left(\frac{-2g\lambda}{\lambda^2}\right)^2 - 2\left(\frac{c}{\lambda^2}\right) \Rightarrow \sum t_i^2 = \left(\frac{2}{\lambda^2}\right)(2g^2 - c)$$

$$\sum (1/t_i)^2 = \left(\sum (1/t_i)\right)^2 - 2\left(\sum (1/t_i t_j)\right)$$

$$= \left(\frac{-2f\lambda}{\lambda^2}\right)^2 - 2\left(\frac{c}{\lambda^2}\right) \Rightarrow \sum (1/t_i)^2 = \left(\frac{2}{\lambda^2}\right)(2f^2 - c)$$

$\Rightarrow$ $\text{Req.} = (2)(2g^2 + 2f^2 - 2c) \Rightarrow \boxed{\text{Req.} = 4r^2}$

375

DATE
PAGE

A) Let n pts (x_i, y_i)

$$\Rightarrow \begin{pmatrix} G \text{ of} \\ \Delta PQS \end{pmatrix} = \left(\frac{x_1+x_2+x_3}{3}, \ \ldots \right)$$

Now, $\sum x_i = 2(3\alpha) = 6\alpha$

$$\sum y_i = 2(3\beta) = 6\beta.$$

$$\Rightarrow \quad G \equiv \left(2\alpha - \frac{x_4}{3}, \ 2\beta - \frac{y_4}{3} \right)$$

$\Rightarrow$ G lies on $(x-2\alpha)^2 + (y-3\beta)^2 = a^2$.

A) Let pts. be $ae^{i\theta_k}$, $k \in \{1, \ldots, 5\}$

By propt, $\left(\frac{z_5}{2}\right) = \left(\frac{a}{4}\right)\left(\sum_{k=1}^{4}\left(e^{i\theta_k}\right)\right)$

where z_5 is centre of hyp. thru $ae^{i\theta_k}$ $k \in \{1, \ldots, 4\}$

Centre of Circle $= 0$.

Now, $$\left[z_5 - \left(\frac{a}{2}\right)\left(\sum_{k=1}^{5} e^{i\theta_k}\right) \right] = \left(\frac{a}{2}\right)\left(e^{i\theta_5}\right)$$

⇒ Centres of hyp. lie on ⊙

$$\boxed{\left| z - \left(\frac{a}{2}\right)\sum_{k=1}^{5}\left(e^{i\theta_k}\right) \right| = \left(\frac{a}{2}\right)}$$